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How Should Inventory Investment be Measured in National Accounts?

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I. Background

Net investment in inventories is one of the more volatile components of GDP, giving it an important role in short-run variation in the growth of GDP and in the timing and duration of economic downturns. Its most distinctive feature may, however, be the number of conceptual and practical difficulties that arise in its measurement both in nominal terms and in real terms.

Net investment in inventories is also known as change in inventories (CII). The first part of this paper concerns the measurement of nominal CII at the annual frequency. The principle that the economic value of production is measured at the prices existing when the production occurred means that the definition of nominal CII must keep holding gains and losses on inventories out of the measure of GDP. Yet, however clear this requirement may be in the realm of theory, it does not translate literally into practice because the data are generally not precise enough to justify complete faith in the raw estimate of holding gains or losses on inventories. If the estimate of holding gains or losses on inventories is large, the implicit price for the annual CII it is likely to be extreme or negative. We wish to avoid the inconvenience that this causes unless we have some assurance that the large estimate is not a result of an unlucky draw of random errors in the underlying data.

In the second half of the paper, we consider the question of how to measure of real investment in inventories. This statistic is important to users of the national accounts because, besides being a component of GDP, inventory investment is an important indicator of economic activity. The usual kind of application of index number theory is impossible for the problem of measuring real CII because a positive change in inventory may be preceded or followed by a zero or negative change in inventory. Besides avoiding division by values that may approach or cross over zero, our proposed measure of real CII appropriately reflects the influence of inventory investment on the measure of real GDP.

II. Measurement of Nominal Change in Inventories

A. Conceptually Correct Treatment of Inventory Holding Gains and Losses

The appropriate definition of nominal CII for a single, detailed item at the annual frequency is a topic of debate in the national accounts literature, with some authors suggesting that either of two approaches can be defended. One approach defines the annual CII as the sum of the twelve nominal monthly CIIs (or four quarterly CIIs if the high-frequency data are by quarter.) The other approach defines annual nominal CII as the annual quantity (or fixed-price) change reflated by the average annual price index for the item.

The two approaches can yield very different results. This is illustrated by the example in table 1, which is simplified by the use of quarterly data instead of monthly data. Summing the quarterly nominal CII estimates in second row of table 1 implies that inventory investment is positive for the year at $87.5 million, yet in the top row of the table the year was characterized by net withdrawals from inventories, resulting in a fall in stocks of 1.9 million tons. Valuing the quantity change at a negative price (as is done when nominal CII for the year differs in sign from the quantity change) is certainly paradoxical, and if the focus were on inventories in isolation, it could potentially be misleading.

Nevertheless, in a context of the national accounts, the objective is to measure production, as summarized by GDP. When current period production of a good exceeds consumption, the excess production is added to inventories, while if consumption is greater than current production, the excess consumption is supplied by a draw down of inventories. Consequently, in the expenditure approach to GDP measurement, production is estimated as final purchases of domestically produced goods and services plus CII.

Production is not only supposed to be recorded at the time that it occurs. It is also supposed to be valued at the price that exists at that time. This is consistent with the principle in national income accounting that holding gains and losses are included neither in measures of production nor in measures of income. Because the price that prevails when a good is produced determines the value of its production, subsequent changes in price during the time that the good is held in inventories are holding gains or losses.[1] Nominal consumption, on the other hand, is recorded when an item is purchased by a final consumer and is valued at the then-current price.[2]

To keep holding gains and losses on inventories out of a measure of production calculated as the sum of consumption and CII, the price that exists when an inventory addition or withdrawal occurs is used to value the addition or withdrawal (SNA93, 6.60, and Foss, *et al.*, no date, 41.) This is in contrast to conventional accounting, which values an item leaving inventory at the same price that it had when it entered inventory. Consequently, in the measurement of nominal GDP using the expenditure approach, CII has a dual role: first, it corrects for timing differences between consumption and production; and second, it corrects for any holding gains and losses occurring while items are held in inventories and realized when they are sold out of inventory as part of the selling price.

The barley producers whose inventory transactions are shown in table 1 had the misfortunate to sell off their inventories when the price was low, only to replenish them when the price was high. Selling when prices are low generates holding losses, and we must remove these

[1] United Nations 1993 System of National Accounts (SNA93), paragraphs 6.58-6.59.
[2] In the national accounts of the US, the mark-up over direct production costs included in the price received is recorded as additional production at the time of sale because inventories are valued at cost of production. The question of whether inventory valuation should include the projected mark-up over direct costs of production is not part of the inventory measurement problem that is the focus of this paper.

holding losses from the recorded sales of the barley producers before we can value their production properly. The correct value of production for our purposes is shown in the Q3 column of table 1 as 564 million dollars. We omit from our illustration any production that was immediately sold rather than added to inventories because we have no information on it.

One of the possible decompositions of the difference between current consumption (i.e. sales) and current production is shown in the top panel of table 2. The quantity of consumption exceeds current year production, so the first step is to exclude an estimate of consumption supplied by production in prior years. A simple proportional split of consumption between current year and prior year production implies that 348 million dollars of consumption were supplied from current production. The price in the quarter when additions to stocks show that production occurred was $109.71. In contrast, the weighted average of prices at which inventories were sold was $67.67. The difference between these prices implies a holding loss of 216 million dollars. Adding this holding loss to consumption of 348 million dollars gives the value of current year production at contemporaneous prices, 564 million dollars.

Provided that annual CII is defined as the sum of the four quarterly CIIs, we can arrive at the same estimate of nominal current production for the year directly by simply adding CII for the year to consumption expenditures—see the second panel of table 2. In contrast, valuing CII at the average annual price would cause the current year's production to be valued at an implausibly low price, as shown in the bottom panel of table 2. Despite the lack of intuitive appeal of the results for the data of table 1, we can identify the SNA93 definition of annual CII as the one that yields the correct measure of current production.

B. A Procedure Suited for a World with Imperfect Data

Despite the inconvenience of discrepancies between nominal CII and the product of the annual price and the annual change in inventory quantities, a definition of annual nominal CII as the sum of the monthly (or quarterly) CIIs would be justifiable as necessary for accurate valuation of current production if we lived in a Platonic world of perfect data. In practice, however, measures of change in inventories are derived from sampled respondents' estimates of opening and closing stocks, each reflecting a varying mix of prices whose composition must be inferred. Both sampling and non-sampling errors are likely in estimates derived in this manner.

Moreover, let us consider more carefully the example of table 1. We had no qualms about valuing the years' production at the Q3 price of $109.71 despite the fact that the average purchaser's price was $67.67, but perhaps we should have. For one thing, in deciding when the production occurred, we failed to consider work-in-progress inventories. They would have revealed that production was not really confined to Q3, so an average of the Q3 price and the Q2 price of $75.52, or $92.62, would be appropriate for valuing the production. Indeed, one could arguably view the production of barley as an annual cycle process, and value the 1988 production at the average price for the year of $85.68.

It seems imprudent to accept large adjustments for holding gains and losses on inventories based on monthly data that are subject to stochastic errors and possible incompleteness, while small adjustments that could be wrong do not seem worth the inconvenience that they cause. The idea of adjusting estimates subject to stochastic errors in the direction of zero has some resemblance to Bayesian shrinkage estimators, which shrink the estimates in the direction of a prior of zero, reducing their variance and their mean squared error (MSE). Indeed, the SNA93 itself concedes—albeit in a paragraph (6.68) that contains two warnings—that data limitations may make “approximate” measures of annual CII more practical than the theoretically ideal one.

Our proposed solution to the problem of measuring inventory change at the item level is a procedure for benchmarking monthly estimates of CII to annual control totals. Measures of non-farm CII in the US national accounts are based on large mandatory surveys at annual frequencies and on smaller voluntary surveys at monthly frequencies. Besides the extra accuracy conferred by the larger sample size of the annual survey, respondents tend to be more careful about the methods used for inventory valuation at year end than they are at other times.

Annual benchmarking is a way to take advantage of the greater reliability of the annual survey data. Once the year is over and measures of annual inventory change have been estimated from the annual survey of inventory stocks, the annual estimates are used to benchmark the original monthly estimates of inventory change.

Three constraints must be satisfied by a benchmarking procedure for monthly CIIs. First, even though the sum of sub-annual nominal CIIs need not equal the annual quantity change times the average annual price, changes in inventory quantities (or fixed-price measures) **are** additive over time. Consequently, we constrain the year-end total of the monthly quantity changes in inventories to equal the benchmark derived from the annual survey. Let q_{it} be the fixed-price (or quantity) CII after benchmarking for inventory item i in month t and let q_i^A be the benchmark for the year of fixed-price CII. Then the quantity adding up constraint is:

$$\text{(C1)} \qquad \sum_{t=1,\ldots,12} q_{it} = q_i^A .$$

Prices are treated as constant within a month, so at the monthly frequency, the nominal CII (c_{it}) is defined to equal the price index (p_{it}) times the fixed-price CII (q_{it}):

(1) $$c_{it} \equiv p_{it} q_{it} \qquad i = 1,2,\ldots,12.$$

The nominal CIIs are used in our second constraint, which follows the conceptual treatment of inventories in the SNA93. This constraint requires that the annual measure of nominal CII equal the sum of the monthly nominal CIIs:

(C2) $$c_i^A = \sum_{t=1,\ldots,12} c_{it}.$$

Third, because no obvious weighting scheme exists for averaging monthly prices, we constrain the annual price index to be the simple average of the monthly price indexes:

(C3) $$\overline{p}_i = \frac{1}{12} \sum_{t=1,\ldots,12} p_{it}.$$

A constraint that could be impossible to satisfy exactly may be maintained as an objective. One such objective is consistency between the annual price, the annual fixed-price CII, and the annual nominal CII, meaning that the implicit price index calculated as the ratio of nominal annual CII to fixed-priced annual CII equals the explicitly calculated annual price index:

(O1) $$c_i^A = \overline{p}_i q_i^A.$$

Imagine for expositional purposes that we begin with estimates of the q_{it} and the p_{it} and solve for three unknowns, q_i^A, c_i^A and $\overline{p}_i$. The solution to the system of three equations represented by (C1), (C2) and (C3) is generally unlikely to satisfy the fourth equation (O1). Alternatively, if we start with the benchmark estimates of q_i^A and c_i^A, then make uniform adjustments to original estimates of the q_{it} to get them to add up to q_i^A and, finally, solve for revised c_{it} via equation (1), we are likely to violate equation (2). Other simple schemes will also lead to frequent violations of one of the four equations.

In the benchmarking process we must adjust the q_{it} to values that add up to q_i^A , but we need not make the same adjustment in every month. Regarding our system of four equations as one in twelve unknowns, rather than three unknowns, transforms the solution set from being empty to containing an infinite number of points. To choose among these points, we solve a minimization problem that embodies another objective: the preservation of information on the intra-year pattern of economic activity and on holding gains and losses. This means that we want the benchmarked monthly fixed-dollar CIIs to have the same pattern over the year as the original estimates of the monthly CIIs derived from the monthly survey data, denoted by q_{it}^M.

We cannot simply impose a binding constraint on the monthly pattern of the benchmarked estimates because doing so would effectively bring us back to a situation where we have four equations in three unknowns, with the likely result of large violations of objective (O1). One reason why objective (O1) is important is that the annual measure of real GDP becomes more consistent with the measure of nominal GDP when the same price and quantity data are used to calculate both measures. In particular, if (O1) is always satisfied, there will be no need to choose between discrepant direct and implicit price indexes for GDP.[3] Second, when (O1) is severely violated—as was the case in the example of table 1—our transformation of an industries' sales (i.e. the consumption of its output) into an estimate of its current production by adding CII incorporates an uncomfortably large adjustment to the price actually realized.

In most cases the intra-year data on inventory change are subject to sampling error and to response error, and the associated prices are also imperfectly measured. We should, therefore, proceed cautiously when adjusting the well-measured consumption value of a good to a quite

[3] Agreement between the direct and implicit price indexes is the most celebrated property of the Fisher index formula used by BEA, but this property requires that the data used in the quantity index be consistent with the data used in the price index. Violations of (O1) by the annual data on change in inventories account for the discrepancy between the published Fisher price index for GDP and the GDP implicit price deflator in the US national accounts.

different estimate of its value in production on the basis of possibly mismeasured data. This is especially so because the inconvenience of gaps between these two values makes our "loss function" asymmetric: an overestimation of the absolute size of this gap is less acceptable than an underestimation. Given these considerations, a desire to take a conservative approach to imputing holding gains or losses on inventories justifies the use of a procedure that adjusts the estimate implied by the raw data to zero if the estimate from the raw data is small, or in the direction of zero if the estimate from the raw data is large.

The solution that implies zero holding gains on inventories is obtained when we impose equation (O1) as a fourth constraint. The Lagrangian for the problem of minimizing the sum of squared differences between the benchmarked monthly estimates q_{it} and the unadjusted estimates q_{it}^{M} subject to (1) the adding-up constraint (C1), and (2) a constraint implied by equation (O1) is:

$$(2) \qquad \min_{q_{i1},\ldots,q_{i,12}} \sum\nolimits_t (q_{it}^{M} - q_{it})^2 + \lambda_1\left\{\left[\sum\nolimits_t q_{it}\right] - q_i^{A}\right\} + \lambda_2\left\{\bar{p}_i\left[\sum\nolimits_t q_{it}\right] - \sum\nolimits_t p_{it}q_{it}\right\}.$$

One way to satisfy the two constraints included in equation (2) is to take advantage of the orthogonality condition $\mathbf{X}'\hat{\mathbf{e}} = \mathbf{0}$ that is a key property of the residuals $\hat{\mathbf{e}}$ from regression on explanatory variables $\mathbf{X}$. This means that the twelve residuals $\hat{e}_{it}$ from a regression of the q_{it}^{M} on the p_{it} and an intercept will sum to zero regardless of whether they are interacted with the p_{it}. This suggests that we might calculate q_{it} as $\hat{e}_{it}$ plus a constant, where $\hat{e}_{it} = q_{it} - \bar{q}_i^{M} + \hat{\beta}_i(p_{it} - \bar{p}_i)$:

$$(3) \qquad q_{it} = \tfrac{1}{12}q_i^{A} + \hat{e}_{it}. \qquad t = 1,2,\ldots,12$$

But is equation (3) the *optimal* solution for the q_{it}? The following proposition answers this question affirmatively.

Proposition 1: The solution to the minimization problem of equation (2) consists of the sums of residuals from a regression of the q_{it}^{M} on the p_{it} and a constant, as shown in equation (3).

Proof: Differentiating equation (2) with respect to the q_{it} shows that:

(4) $$q_{it} = q_{it}^{M} + \lambda_1/2 + (\lambda_2/2)(p_{it} - \overline{p}_i) \qquad t = 1,2,\ldots,12.$$

Equation (4) shows that a change in λ_1 affects each of the q_{it} by the same constant amount. In particular, replacing q_i^A inside the λ_1 constraint in equation (2) by 0 will change only the value of λ_1, so after we adjust it by adding the appropriate constant (its value is $\frac{1}{12}q_i^A$), the solution for q_{it} from an alternative problem where the q_{it} are constrained to add up to 0 is the same as the solution equation (2). Therefore, provided that we remember to add the constant $\frac{1}{12}q_i^A$ to the solution vector, we can treat the problem in equation (2) as equivalent to a problem where the objective is to minimize $(\mathbf{y}-\hat{\mathbf{e}})'(\mathbf{y}-\hat{\mathbf{e}})$ subject to the constraints given by the rows of the equation $\mathbf{X}'\hat{\mathbf{e}} = \mathbf{0}$, where $\mathbf{y}$ is the vector of the q_{it}^{M}, and $\mathbf{X}$ is an explanatory variable matrix with 1s in its first column and the p_{it} in its second column. Substituting in the definition for $\hat{\mathbf{y}}$ implicit in the identity $\mathbf{y} \equiv \hat{\mathbf{y}} + \hat{\mathbf{e}}$, we can rewrite this objective as one of minimizing the "regression sum of squares" $\hat{\mathbf{y}}'\hat{\mathbf{y}}$ subject to $\mathbf{X}'\hat{\mathbf{y}} = \mathbf{X}'\mathbf{y}$. Letting $X_{1t} = 1$ for all t to reflect presence of 1s in first column of $\mathbf{X}$ and letting $\mathbf{X}$ have K columns for the sake of generality, the solution is found by minimizing the Lagrangian containing the constraints given by the rows of $\mathbf{X}'\hat{\mathbf{y}} = \mathbf{X}'\mathbf{y}$:

(5) $$\min_{\hat{y}_1,\ldots,\hat{y}_T} \sum_t \hat{y}_t^2 + \lambda_1\Big[\sum_t (y_t - \hat{y}_t)\Big] + \lambda_2\Big[\sum_t X_{2t}(y_t - \hat{y}_t)\Big] + \ldots + \lambda_K\Big[\sum_t X_{Kt}(y_t - \hat{y}_t)\Big]$$

The derivatives of this Lagrangian with respect to the $\hat{y}_t$ imply that each $\hat{y}_t$ is the *same* linear function of the explanatory variables (specifically, the function whose coefficients are $\lambda_1/2, \lambda_2/2,\ldots,\lambda_K/2$.) Denoting the vector of these coefficients by $\hat{\beta}$, substitute $\mathbf{X}\hat{\beta}$ for $\hat{\mathbf{y}}$ in the constraint $\mathbf{X}'\hat{\mathbf{y}} = \mathbf{X}'\mathbf{y}$ and solve for $\hat{\beta}$. This yields the standard regression formula of $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$, which establishes that the $\hat{\mathbf{e}}$ that solves our minimization problem is indeed the standard regression residual vector. □

Equation (3) is quite satisfactory as an estimator for the fixed-price monthly inventory change when R^2 from the regression of the q_{it}^M on the p_{it} is not large: in this case, the benchmarked monthly CIIs will have a pattern over the year that closely mimics the original monthly CIIs. (For convenience, we omit the *i* subscript on R^2 even though it obviously varies across items.) If, on the other hand, the regression's R^2 is near 1, the residuals will be near zero and the q_{it} will preserve little of the original pattern of the q_{it}^M. The undesirability of high R^2s makes our problem quite unlike the usual applications of the regression technique, where the research hopes for a high R^2.

If the monthly data strongly suggest the presence of significant holding gains or losses on inventories and of large differences between months in net inventory investment, these patterns are unlikely to arise entirely from incomplete or inaccurate data. Under this circumstance, therefore, the annual estimate of nominal CII should provide for the presence in the consumption data of some holding gains or losses the need to be kept out of the measure of output.

To make sure that most of the original pattern of the monthly quantities is retained after benchmarking in cases when R^2 is high, in these cases we include a damping factor *k* that moves

$\hat{\beta}_i$ in the direction of zero. Let R^{2*} denote the maximum allowed for the proportion of the variance of the original monthly CIIs removed in benchmarking. This is equivalent to requiring that the benchmarked monthly CIIs preserve enough of the variation originally present to have a variance equal to at least $1 - R^{2*}$ times the original variance. In the appendix we show that to achieve this, the damping factor should be calculated as:

$$k = 1 - [1 - R^{2*}/R^2]^{1/2} \tag{6}$$

if $R^2 > R^{2*}$. Letting $k = 1$ if $R^2 \leq R^{2*}$ and letting $\hat{\beta}_i$ be the regression slope coefficient on the p_{it}, the generalized expression for the residual is:

$$\tilde{\varepsilon}_{it} = q_{it}^M - [\, \bar{q}_i^M + k\hat{\beta}_i(p_{it} - \bar{p}_i) \,]. \tag{7}$$

The benchmarked fixed-dollar monthly CIIs that include the damping factor are defined as

$$\tilde{q}_{it} = \tilde{\varepsilon}_{it} + q_i^A/12. \qquad t = 1,2,\ldots,12. \tag{8}$$

Note that $\sum_t \tilde{q}_{it} = q_i^A$, so constraint (C1) is still satisfied. However, for $R^2 > R^{2*}$, the benchmarked estimate of current-dollar annual CII no longer equals $\bar{p}_i q_i^A$. Substituting the formula for the regression slope coefficient $\mathrm{Cov}(\mathbf{p}_i,\mathbf{q}_i)/\mathrm{Var}(\mathbf{p}_i)$ for $\hat{\beta}_{i2}$ in the equation for $\tilde{\varepsilon}_{it}$ shows that $\sum_t p_{it}\tilde{\varepsilon}_{it} = (1 - k)\mathrm{Cov}(\mathbf{p}_i,\mathbf{q}_i)$, so:

$$\sum_t p_{it}\tilde{q}_{it} = (1 - k)\mathrm{Cov}(\mathbf{p}_i,\mathbf{q}_i) + \bar{p}_i q_i^A. \tag{9}$$

Assuming that the monthly inventory change data are perfectly measured, the value of the holding gains embedded in the annual sales figures for item i is $-\mathrm{Cov}(\mathbf{p}_i,\mathbf{q}_i)$ and annual current-dollar output of item i equals its annual consumption plus net investment in inventories measured as Cov($\mathbf{p}_i$,$\mathbf{q}_i$) + $\bar{p}_i q_i^A$. Therefore *k* equals the proportion of the holding gains on inventories implied by the monthly data that is **not** removed from the benchmarked annual measure of output when nominal net investment in inventories for the year is measured by equation (8).

In practice, we set R^{2*} equal to 0.333, so that at least two-thirds of the variance in the original fixed-dollar monthly CIIs is always preserved. Under the null hypothesis of no systematic relationship between prices and quantities, we can expect to see $R^2 \leq 0.333$ about 95 percent of the time assuming normally distributed errors.

C. Problem caused by Requiring Nominal GDP to Equal Real GDP in the Base Year

The convention of setting nominal GDP equal to fixed-price or real GDP in the base year is usually viewed as an innocent, mathematically trivial normalization. This would indeed be true had we defined the annual nominal measure of inventory change for a detailed item as a reflated fixed-price measure. As that is not the case, to obtain equality in the base year between components of nominal GDP and corresponding components of real GDP, we must choose between a distortion in the base year estimate of nominal GDP and a distortion in the estimate of the change in real GDP. Our one consolation in making this unpleasant choice is that neither distortion is likely to be numerically important.

The genesis of the problem is the use of the measure of nominal CII to adjust for the holding gains and losses on inventory items that do not belong in the measure of production. For non-inventory items, the annual price can be calculated as a unit value (i.e. a quantity-weighted average), ensuring equality between the nominal expenditure for the year and the product of the annual price and the annual quantity. In contrast, for inventory items, no quantity weights are available, so a simple average of monthly prices must be used. Consequently, the product of the annual price and the annual quantity fails to equal the annual nominal CII in cases where $k < 1$, as shown in equation (9).

At least a few such cases are likely to occur in the base year. In these cases, to obtain identical measures of nominal and fixed-price CII, we must either adjust the measure of nominal CII or the measure of fixed-price CII. If we choose the former strategy, making the measure of nominal CII equal to q_i^A, the measure of nominal GDP in the base year will be distorted. In effect, the wrong price will be used to value production. For example, with the data of table 1, barley production would be valued at \$61.02 per ton, far below the range of theoretically defensible prices of \$92.61 to \$109.71.

If, on the other hand, we adjust the estimate of fixed-price annual CII to make it equal to the nominal annual CII of $\sum_t p_{it}\tilde{q}_{it}$, the inclusion of holding gains and losses in the measure of fixed-price CII in the base year will distortion the estimates of change in real CII. For example, given the data in table 1, the estimator for nominal annual CII on the left side of equation (9) would imply an adjusted estimate of fixed-price CII of about +\$5 million, so that the negative physical change in inventory stocks was valued at negative price. Furthermore, if the measure of fixed-price inventory stocks is built up by cumulating fixed-price inventory investment (as is the case in the NIPAs) the estimates of fixed-price stocks will also be distorted if the base year estimate of fixed-price investment is adjusted to equal the nominal investment.

The distortion in the estimate of nominal GDP is more acceptable than the one in the estimated growth of real GDP, so in the base year we require that nominal CII always equal q_i^A, where the units of measurement for q_i^A are such that $\overline{p}_i = 1$ in the base year. This could easily be accomplished by letting k equal 1 in equation (7) even when R^2 is high, but the result would be to compound the error by introducing a second distortion, an excessive smoothing of the intra-year pattern of CII. (A regression R^2 near 1 implies that the residuals are near zero, so when we use these residuals to benchmark the months, any pattern present in the monthly data will be revised to a flat line.) We therefore calculate k in the usual manner in the base year, letting it be small when R^2 is large. We then eliminate the discrepancy that occurs in these cases between q_i^A (the benchmark value of CII for base year given that in that year $\bar{p}_i \equiv 1$) and $\sum_t p_{it}\tilde{q}_{it}$ by distributing it over the months in proportion to the absolute values of the monthly CIIs. This minimizes the distortion to the relative magnitude of the monthly nominal CIIs (i.e. the $p_{it}\tilde{q}_{it}$). In the base year the monthly nominal CII for item i is calculated as:

$$\tilde{c}_{it} = p_{it}\tilde{q}_{it} + \frac{|p_{it}\tilde{q}_{it}|}{\sum_s |p_{is}\tilde{q}_{is}|}[q_i^A - \sum_s p_{is}\tilde{q}_{is}]. \tag{10}$$

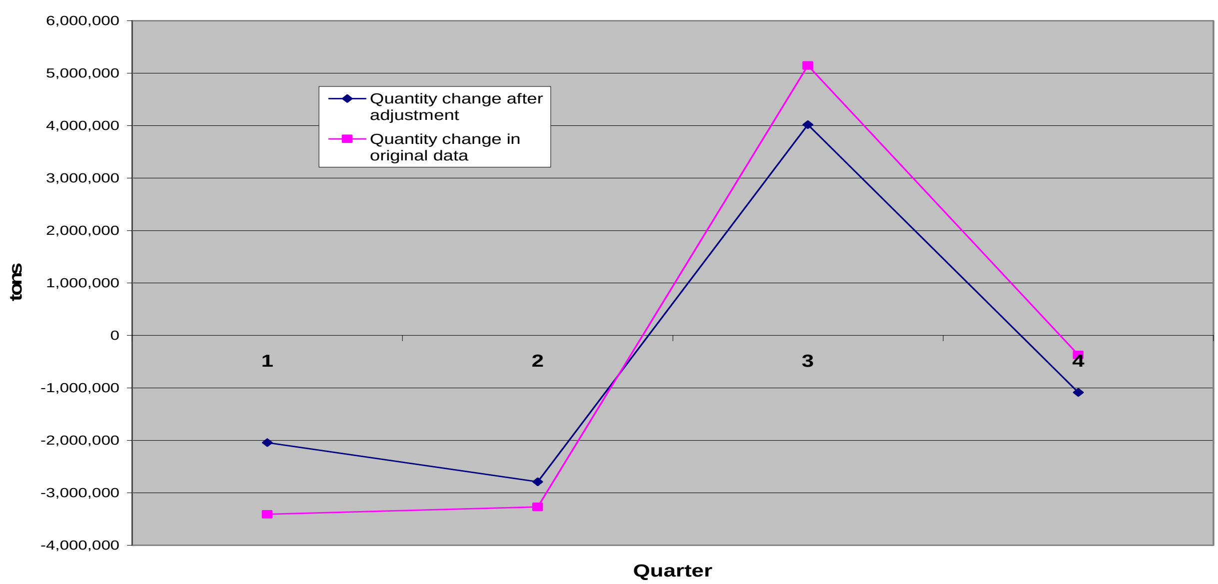
For items with $k < 1$ in the base year, the second term on the right side of equation (10) will be non-zero. As a result, the monthly nominal CII will no longer equal the product of the monthly price and the monthly fixed-price CII. The monthly counterpart of constraint (C1) is then violated in order to preserve information on the intra-year pattern of the CII in the base year that would otherwise be lost.

D. Empirical Tests of the Benchmarking Method

We begin our empirical tests of the methods proposed above with the Canadian barley data of table 1. Although the example was not one of benchmarking to a predetermined annual total, for illustrative purposes we assume that there was an annual benchmark equal to the actual annual total of -1,902,051 tons.

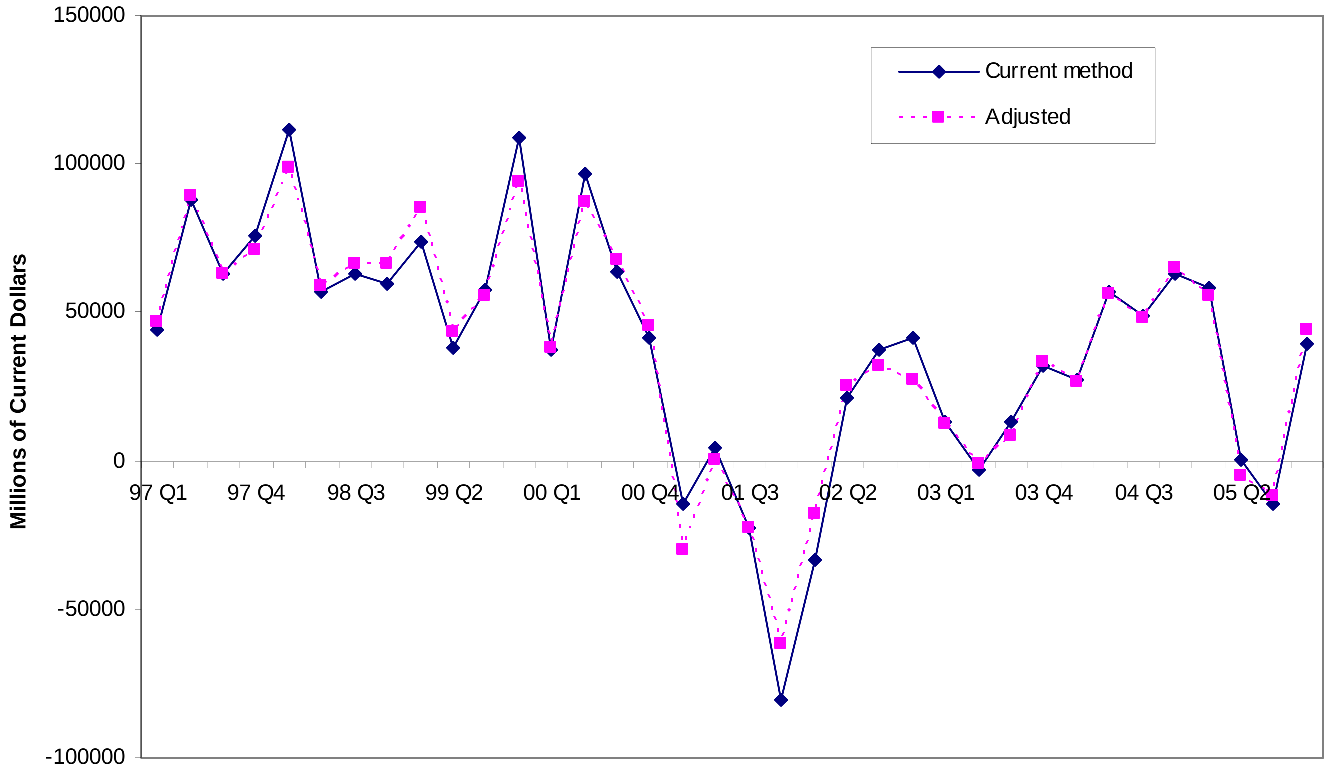
Not surprisingly, in the case of the data in table 1, R^2 from the regression of quantities on prices is quite high, with a value of 0.742. Our damping factor k from equation (6) therefore equals 0.33. This leads to relatively modest revisions of the unadjusted quantity data, as is seen in figure 1. The revised quantities imply revised nominal CIIs for the four quarters that yield a downward revision of the estimate of nominal annual CII from $87,541 to $4,952—see table 3. The implied nominal measure of annual production has the same downward revision. The revised price that is implicitly being used to value the production may be found by dividing the revised nominal measure of production of $481,839,000 by the quantity of production of 5,144,939 tons, resulting in a price of $93.65. This is quite close to the average of the Q2 and Q3 prices of $92.61, which above we found to be more reasonable than price of $109.71 that is implied by unadjusted annual measure of nominal CII. Thus, equation (8) seems to perform satisfactorily when confronted with the Canadian barley data.

Figure 1: Adjusted Quantity Change for Example of Barley in Canada in 1988



We also conducted extensive empirical tests of the equation (8) method using U.S. national income and product accounts (NIPA) data for the time period 1997 to 2005. For this time period, the NIPA data on changes in inventory are calculated on an industry basis using the

Figure 2: Quarterly Current-Dollar Changes in Inventory, Item no. CBI_Nonfarm



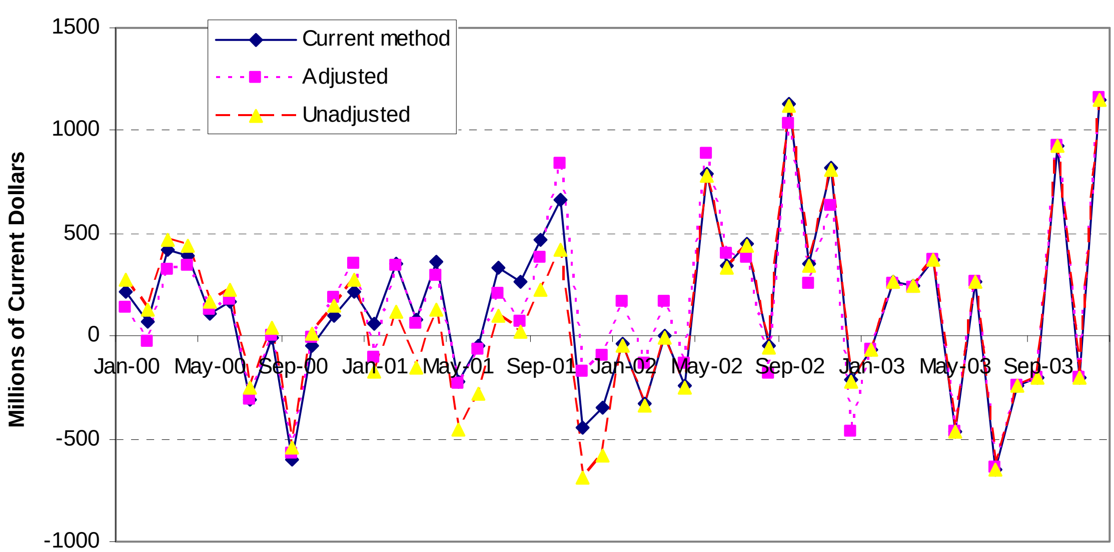
North American Industry Classification System (NAICS). Annual estimates of change in inventory are calculated for 568 industries (417 industries starting in 2002). Of those industries, 473 represent the manufacturing sector (322 industries starting in 2002). Each manufacturing industry comprises three subcategories, one for each stage of fabrication (materials and supplies, work-in-progress, and finished goods). Therefore, a total of 1,514 individual industry series have to be benchmarked each year.

Equation (8) is not currently used by BEA, so in the tests we were interested in both the revisions to the unadjusted data and how the series that were adjusted using equation (8) compared to those calculated using the current method. (The currently used method achieves satisfaction of constraint (O1) by adding the necessary constant to each of the unadjusted monthly measures of nominal CII.)

A modest reduction in volatility in the estimate of aggregate current-dollar changes in nonfarm inventory when equation (8) is used is evident in figure 2, which charts the results. In general, though, the two methods yield similar results. One likely cause of the similar results is that the correlation between monthly prices and monthly fixed-price CIIs is low in most cases.

In cases with an R^2 of 0.333 or less, we are able to satisfy all the constraints by setting *k* equal to 1 in equation (7). An example of an industry where all constraints were always satisfied is merchant wholesale motor vehicles (item no. cbi4211), shown in figure 3. However in one year, 2001, R^2 for this industry was a bit high at 0.21 and the annual benchmark implied the need for a substantial upward revision in the average level of the monthly estimates. The price index fell sharply in the fourth quarter, so that is where equation (8) concentrated the revision.

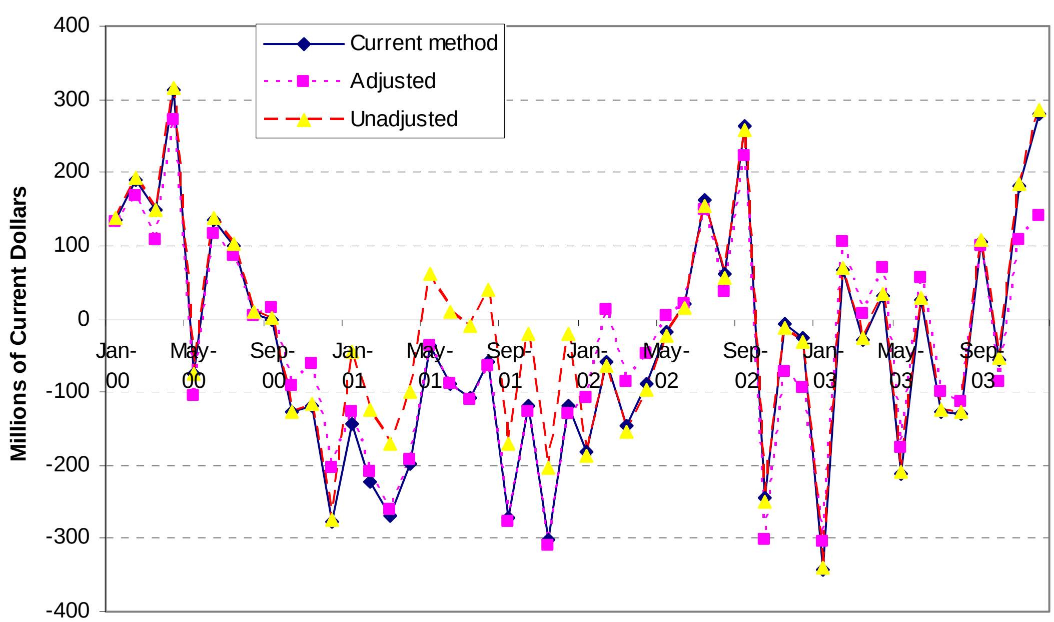
Figure 3: Monthly Current-Dollar Changes in Inventory, Item no. cbi4211



In a few cases, objective (O1) could not be satisfied because doing so would result in too much loss of information on the monthly pattern of inventory investment and on inventory holding gains or losses. In 2000, the year of the highest incidence of such cases, objective (O1) was relaxed 4.8 percent of the time, but in 1997 it was relaxed just 1 percent of the time.

The merchant wholesale metal and minerals industry (item no. cbi4215) furnishes examples of the relaxation of objective (O1) in the years 2000 and 2003—see figure 4. In those years, the benchmarked estimate of annual nominal CII differs from the product of the annual price and the annual fixed-price CII because it is calculated as the sum of adjusted monthly nominal CIIs that are not forced to be consistent with a measure of annual CII calculated as the product of the annual price index and the annual fixed-price CII. The effect of this is most noticeable in late 2003, when a divergence is seen between the pattern of the adjusted series and the (shared) pattern of the unadjusted series and the current method series.

Figure 4: Monthly Current-Dollar Changes in Inventory, Item no. cbi4215



III. Measuring Real Change in Inventories in a Fisher Index Framework

We now shift our focus from the measurement of CII for a single item to the measurement of aggregate CII and from the measurement of nominal values to the measurement of real values. Once the question of how to form nominal measures of CII for individual items has been settled, aggregating them is a trivial matter. In contrast, designing an aggregate measure of real CII is a difficult problem. A Laspeyres or Paasche index calculated in the usual way from CII data may have a numerator that differs in sign from its denominator, or a denominator that is near zero. Under these circumstances, standard index numbers lose their meaningfulness, suggesting that the attempt to measure aggregate real CII is futile (Diewert, 2004, 2005.)

Yet even if it were true that a meaningful measure of real CII can sometimes fail to exist, that would not be a reason to deprive users of the national accounts of information that is generally useful by dropping real CII from the published set of economic indicators. We therefore seek a solution to the problem of measuring real CII that is applicable under the broadest possible range of circumstances.

In the NIPAs, the solution to the need to avoid indexes of inventory flows has been to calculate real CII as the first difference of real inventory *stocks*. This procedure uses stocks as weights for flows, ignoring the fact that the composition of the stocks may differ greatly from the composition of the inventory flows. A theoretically superior procedure for estimating real CII that subtracts real gross disposals from real gross inventory acquisitions was, therefore, developed by Ehemann (2005). That method is quite data-intensive, however, and data on gross flows into and out of stocks may even be unavailable in some cases. Hence, we seek a method of measuring real CII that avoids the need for separate data on inventory gross flows.

A. Inventories in the Calculation of the Fisher Index Measure of Real GDP

In the base period, assumed to be period *t*–1, real GDP is set equal to current-dollar GDP. In subsequent time periods, the chain Fisher index measure of real GDP in any time period *s* after the base period is constructed by multiplying the previous value of real GDP by the Fisher quantity index from period *s*–1 to period *s*. In time periods preceding the base period, real GDP is measured by dividing by a Fisher quantity, starting with the one from period *t*–2 to period *t*–1.

The procedure can be illustrated by the first link in the forward chain, beginning with a simplifying assumption that all inventory changes equal zero. Let $p^f_{ct}\, q^f_{ct}$ represent the current-price final consumption of commodity *c*, and let $p^f_{c,t\text{-}1}\, q^f_{ct}$ represent the value of these expenditures deflated to period *t*–1. With no changes in inventories, nominal GDP in period *t* equals $\sum_c p^f_{ct} q^f_{ct}$. The Fisher quantity index of final demand, F^f_t, is:

(11) $$F^f_t \;=\; \left[\frac{\sum_c p^f_{ct\text{-}1}\, q^f_{ct}}{\sum_c p^f_{ct\text{-}1}\, q^f_{ct\text{-}1}}\;\frac{\sum_c p^f_{ct}\, q^f_{ct}}{\sum_c p^f_{ct}\, q^f_{ct\text{-}1}}\right]^{0.5},$$

Real GDP measured in chain-dollars of period *t*–1 would equal $(\mathbf{p}^{f}_{t\text{-}1}\cdot\mathbf{q}^{f}_{t\text{-}1})\, F^f_t$.

To bring changes in inventories into the calculation of real GDP, let q_{it} be the value in prices of base period b of the net additions to inventories of inventory item *i*. Although the period length here is a year, it is convenient to change the notation for the price index from $\bar{p}_i$ to PA_{it}, where PA_{it} is the price index appropriate for deflating the flows (which are effectively assumed to occur at uniform rate) during period t. Nominal net investment in inventory in period *t* is, then, $\sum_i PA_{it}\, q_{it}$, or $\mathbf{PA}_t\cdot\mathbf{q}_t$. Nominal GDP in period *t* is $N^{GDP}_t = \sum_c p^f_{ct} q^f_{ct} + \sum_i PA_{it}\, q_{it}$. Real GDP in dollars of period *t*–1, denoted Q^{GDP}_t, is:

(12) $$Q_t^{GDP} = N_{t-1}^{GDP} \left[\frac{\mathbf{p}^f_{t-1}\cdot \mathbf{q}^f_t + \mathbf{PA}_{t-1}\cdot \mathbf{q}_t}{\mathbf{p}^f_{t-1}\cdot \mathbf{q}^f_{t-1} + \mathbf{PA}_{t-1}\cdot \mathbf{q}_{t-1}} \frac{\mathbf{p}^f_t\cdot \mathbf{q}^f_t + \mathbf{PA}_t\cdot \mathbf{q}_t}{\mathbf{p}^f_t\cdot \mathbf{q}^f_{t-1} + \mathbf{PA}_t\cdot \mathbf{q}_{t-1}} \right]^{0.5}.$$

None of terms in equation (12) can be identified as a measure of real CII; the four terms containing CII are just deflated (or reflated) measures of fixed-price CII in period t or period t–1. The reason to estimate real CII is, then, to have a statistic that summarizes the inventory quantity changes occurring in the economy, not to have a building block for calculating real GDP. Nonetheless, we would like our measure of real CII to be sufficiently related to the measure of real GDP so that it reflects the influence on real GDP of inventory investment.[4] This influence depends on the value of the change in CII between times t–1 and t at prices of time t–1 and of time t. In particular, if final purchases are constant, the difference between the numerator and denominator of the quantity index for GDP equals $\mathbf{PA}_{t-1}\cdot(\mathbf{q}_t - \mathbf{q}_{t-1})$ in the Laspeyres case, and $\mathbf{PA}_t\cdot(\mathbf{q}_t - \mathbf{q}_{t-1})$ in the Paasche case. This suggests that difference equations may be a promising alternative to the unusable index number methods.

B. The Method of First-Differencing Stocks

In the NIPAs, the fixed-dollar stock of any inventory item i is calculated by accumulating fixed-dollar inventory changes. Thus, the value of this stock at the end of period t, denoted by k_{it}, equals $k_{i,t-1} + q_{it}$. The aggregate fixed-dollar stock at the end of period t–1 equals $\sum_i k_{i,t-1}$. The current-dollar stock of inventories at the end of period t–1 is measured using end-of-period price indexes PK_{it}, where PK_{it} is related to PA_{it} by the equation $PA_{it} = \frac{1}{2}(PK_{i,t-1} + PK_{it})$. The

[4] The objective of reflecting the role of inventory investment in real GDP may mean that we must forego another objective, measuring the evolution of the aggregate real stocks.

Laspeyres index of the stock of inventories at the end of month t compared with the stock at the end of month *t*–1 is:

(13) $$L_t^K = \frac{\mathbf{PK}_{t\text{-}1}\cdot \mathbf{k}_t}{\mathbf{PK}_{t\text{-}1}\cdot \mathbf{k}_{t\text{-}1}}.$$

Similarly, the Paasche index of the change in the inventory stock is:

(14) $$P_t^K = \frac{\mathbf{PK}_t\cdot \mathbf{k}_t}{\mathbf{PK}_t\cdot \mathbf{k}_{t\text{-}1}}.$$

Finally, the Fisher stock index F_t^K equals $[L_t^K P_t^K]^{0.5}$.

Measuring real CII as the change in the real stocks, results in an estimate for period *t* (the first period after the base period) measured in end-of-period dollars from time *t*–1 of:

(15) $$Q_t = (\mathbf{PK}_{t\text{-}1}\cdot \mathbf{k}_{t\text{-}1})(F_t^K - 1).$$

The NIPA measure of chain-dollar real CIPI, Q_t, valued in dollars from the middle of period t–1 equals $(\mathbf{PA}_{t\text{-}1}\cdot \mathbf{PK}_{t\text{-}1} / \mathbf{PK}_{t\text{-}1}\cdot \mathbf{k}_{t\text{-}1})Q_t$, or $(\mathbf{PA}_{t\text{-}1}\cdot \mathbf{k}_{t\text{-}1})(F_t^K - 1)$.

In principle, the period *t*–1 measure of real CII should differ from the period *t*–1 measure in current dollars because it should equal $(\mathbf{PA}_{t\text{-}1}\cdot \mathbf{k}_{t\text{-}1})(1 - 1/F_{t\text{-}1}^K)$, whereas the corresponding nominal value of CII is $\mathbf{PA}_{t\text{-}1}\cdot \mathbf{q}_{t\text{-}1}$. In practice, however, a difference between nominal and real CII in the base year is unacceptable. Such a difference is avoided by introducing a special link equal to $\mathbf{PA}_{t\text{-}1}\cdot \mathbf{q}_{t\text{-}1}$ for the base year CII.

C. Problems with the Method of First-Differencing Stocks

To analyze the stocks-based measure of real CII, note that:

(16) $$[P_t^K]^{0.5} L_t^K + [L_t^K]^{0.5} P_t^K = [L_t^K]^{0.5} [P_t^K]^{0.5} \left\{ [L_t^K]^{0.5} + [P_t^K]^{0.5} \right\}.$$

Dividing both sides of the equation by the final bracketed term shows that the Fisher index is a weighted arithmetic average of the Laspeyres and Paasche indexes, with the weight on the Paasche index proportional to the square root of the Laspeyres index, and the weight on the Laspeyres index proportional to the square root of the Paasche index. We can therefore express the Fisher index as:

$$F_t^K = \lambda_t L_t^K + (1-\lambda_t) P_t^K \tag{17}$$

where

$$\lambda_t = \frac{[P_t^K]^{0.5}}{[L_t^K]^{0.5} + [P_t^K]^{0.5}} . \tag{18}$$

Another way to express λ_t is as the ratio of the Fisher index to the sum of the Fisher and Laspeyres indexes. It is likely to be very near 0.5 for the indexes used to measure real inventory change, because using stocks as weights is likely to result in close agreement between the Laspeyres index and the Paasche index.[5]

Let L_t^{KP} denote the Laspeyres price index that uses stocks for weights, or $\mathbf{PK}_t \cdot \mathbf{k}_{t-1} / \mathbf{PK}_{t-1} \cdot \mathbf{k}_{t-1}$. Substituting from equation (4) for F_t^K in equation (6), and substituting $\mathbf{q}_t$ for $\mathbf{k}_t - \mathbf{k}_{t-1}$, after some algebra we find that the measure of real CII is an average of fixed-dollar change in inventories measured in period t–1 dollars, and current-dollar change in inventories deflated by the stock-weighted Laspeyres price index:

$$Q_t = (\mathbf{PK}_{t-1} \cdot \mathbf{k}_{t-1})\left[F_t^K - 1\right] \tag{19}$$

$$= (\mathbf{PK}_{t-1} \cdot \mathbf{k}_{t-1})\left[\lambda_t L_t^K + (1-\lambda_t) P_t^K - 1\right]$$

[5] The weight on the Laspeyres index in equation (17) differs from 0.5 by an order of magnitude less than the proportion by which the Laspeyres index exceeds the Paasche index. For example, if the Laspeyres-to-Paasche ratio is 1.01, then the weight for the Laspeyres index is about 0.499.

$$= \lambda_t \mathbf{PK}_{t\text{-}1}\cdot\mathbf{k}_t + (1-\lambda_t)\frac{\mathbf{PK}_t\cdot\mathbf{k}_t}{L_t^{KP}} - \mathbf{PK}_{t\text{-}1}\cdot\mathbf{k}_{t\text{-}1}$$

$$= \lambda_t \mathbf{PK}_{t\text{-}1}\cdot\mathbf{q}_t + (1-\lambda_t)\frac{\mathbf{PK}_t\cdot\mathbf{k}_t}{L_t^{KP}} - (1-\lambda_t)\mathbf{PK}_{t\text{-}1}\cdot\mathbf{k}_{t\text{-}1}$$

$$= \lambda_t \mathbf{PK}_{t\text{-}1}\cdot\mathbf{q}_t + (1-\lambda_t)\frac{\mathbf{PK}_t\cdot\mathbf{k}_t}{L_t^{KP}} - (1-\lambda_t)\frac{\mathbf{PK}_t\cdot\mathbf{k}_{t\text{-}1}}{L_t^{KP}}$$

$$= \lambda_t \mathbf{PK}_{t\text{-}1}\cdot\mathbf{q}_t + (1-\lambda_t)\frac{\mathbf{PK}_t\cdot\mathbf{q}_t}{L_t^{KP}}\,.$$

In equation (19), the weights in the deflator for the current-dollar estimate of CII, L_t^{KP}, have no direct relationship to the composition of $\mathbf{q}_t$. As a result, Q_t cannot be relied on to yield a sensible measure of real CII. In particular, it can violate even a weak monotonicity axiom that requires that if every element of $\mathbf{q}_t$ is larger than the corresponding element of $\mathbf{q}_{t\text{-}1}$, Q_t must be larger than $Q_{t\text{-}1}$.[6] Furthermore, a comparison of Q_t with $Q_{t\text{-}1}$ may give a misleading indication of the effect of CII on real GDP. Ehemann (2005) demonstrates the empirical importance of this problem in tests with data from the NIPAs.

To generalize equation (19) to cover any time period, let CF_s^{KP} be the value in the arbitrary year s of the chained Fisher price index that uses stocks as weights. Then difference in real stocks is a mean of two measures of deflated CII in year s, one based on beginning-of-year prices and another based on end-of-year prices times a scalar that approximately equals 1. (The scalar’s exact value is $2\lambda_s$.) In both measures, the weights reflect the composition of the stocks, not the flows.

(20) $$Q_s = \lambda_s(\mathbf{PK}_{s\text{-}1}/CF_{s\text{-}1}^{KP} + \mathbf{PK}_s/CF_s^{KP})\cdot\mathbf{q}_s\,.$$

[6] Arnold Katz has constructed numerical examples that demonstrate this failure of weak monotonocity.

D. An Improved Measure for Real CII

Although equation (19) appears to be an unsatisfactory way to measure real CII, its functional form does suggest an approach that avoids an application of index number techniques to data that are unsuitable for this purpose. That approach is based on the observation that the value of one year's quantity changes at its own or some other year's prices is always a well-defined concept, regardless of its suitability for index number construction. Thus, for example, we ask can how much the changes in quantities from last year to this year would be worth at last year's prices, and also how much these changes would be worth at this year's prices. We can then average these two dollar-denominated measures, with just a limited reliance on a price index for deflation purposes.

To develop a measure of real CII based on a sum of terms that measure the value of $\mathbf{q}_t$ at prices from periods t–1 and t, we use the axiomatic (or test) approach to index numbers as a guide. Four axioms that the improved measure of real CII, denoted by Q_t^*, should ideally satisfy are listed below. The first three of these axioms each imply the use of the inventory flow prices $\mathbf{PA}_t$, so we no longer consider the use of the inventory stock prices $\mathbf{PK}_t$.

Axiom I: Generalization of the ordinary measure of real values in the NIPAs. The need for special procedures for measuring real changes in inventories arises because the changes may not all have the same sign. However, in cases where all the values do have the same sign, the standard Fisher measure is known to have many advantages. If all the changes in inventories have the same sign, these advantages should not be sacrificed unnecessarily, and if the changes are predominantly of one sign, the measure should come close to the one that has these advantages. Therefore, if $q_{i,t\text{-}1} \geq 0$ and $q_{it} \geq 0$ for all i, with the inequality strict for least one $q_{i,t\text{-}1}$ and at least one q_{it}, then Q_t^* should equal the ordinary measure of real chain-dollar value, defined as $Q^{*}_{t-1}F^{q}_{t}$, where F^{q}_{t} is the geometric mean of the Laspeyres index for flows, $\frac{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t}}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}$, and the Paasche index for flows, $\frac{\mathbf{PA}_{t}\cdot\mathbf{q}_{t}}{\mathbf{PA}_{t}\cdot\mathbf{q}_{t-1}}$, and where $Q^{*}_{t-1} \equiv \mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}$.

Axiom II: Approximately monotonic relationship to effect on real GDP. If $\mathbf{q}^{*}_{t}$ implies a value for Q^{GDP}_{t} that exceeds the value implied by $\mathbf{q}_{t}$ by more than some small positive amount Δ, then Q^{*}_{t} should be larger when calculated with $\mathbf{q}^{*}_{t}$ than when calculated with $\mathbf{q}_{t}$. A positive tolerance Δ is necessary in this axiom because the effect of CII on real GDP partly depends on the variables entering the calculation of real GDP but not the formula for real CII.

Axiom III: Sign agreement with any effect on real GDP of non-trivial magnitude. The change in real CII should have the same sign as the contribution of the change in CII to real GDP if the latter differs from 0 by a non-trivial amount. That is, the sign of the effect on real GDP of the change from $\mathbf{q}_{t-1}$ to $\mathbf{q}_{t}$ should be the same as the sign of $Q^{*}_{t} - Q^{*}_{t-1}$ if $|Q^{*}_{t} - Q^{*}_{t-1}| > \Delta$ for some $\Delta > 0$. Sign agreement is, of course, implied by satisfaction of the monotonicity axiom, because in that axiom $\mathbf{q}_{t}$ can equal $\mathbf{q}_{t-1}$.

Axiom IV: Homogeneity of degree 0 in final period prices. If every price $\text{PA}_{i,t-1}$ is multiplied by the same constant to obtain $\mathbf{PA}_{t}$, the structure of relative prices is unchanged and the chain-dollar measure of real CIPI should equal the fixed-dollar measure $\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t}$. More generally, if for some positive constant k, $\mathbf{PA}^{*}_{t} = k\mathbf{PA}_{t}$, then the substitution of $\mathbf{PA}^{*}_{t}$ for $\mathbf{PA}_{t}$ in the formula for Q^{*}_{t} should have no effect on the result.

Proposition 2 introduces a Fisher-like measure of real CII that satisfies our axioms.

Proposition 2. Let $\mathbf{q}_t^{ds}$ denote the vector of "dominant-sign" elements $\mathbf{q}_t$, where zeros replace the negative elements of $\mathbf{q}_t$ if $\mathbf{PA}_t \cdot \mathbf{q}_t \geq 0$ or the positive elements of $\mathbf{q}_t$ if $\mathbf{PA}_t \cdot \mathbf{q}_t < 0$. Let L_t^{ds} equal the Laspeyres price index $\frac{\mathbf{PA}_t \cdot \mathbf{q}_{t-1}^{ds}}{\mathbf{PA}_{t-1} \cdot \mathbf{q}_{t-1}^{ds}}$ if at least one element of $\mathbf{q}_{t-1}$ is non-zero or, if the Laspeyres index is undefined, let L_t^{ds} equal the analogous Paasche price index $\frac{\mathbf{PA}_t \cdot \mathbf{q}_t^{ds}}{\mathbf{PA}_{t-1} \cdot \mathbf{q}_t^{ds}}$. Similarly, let P_t^{ds} equal the Paasche price index or, if the Paasche index is undefined, let P_t^{ds} equal the Laspeyres price index. If either $\mathbf{q}_t$ and $\mathbf{q}_{t-1}$ has at least one non-zero element, μ_t is defined as:

$$\mu_t = \frac{[P_t^{ds}]^{0.5}}{[L_t^{ds}]^{0.5} + [P_t^{ds}]^{0.5}} . \tag{21}$$

Otherwise, let $\mu_t = ½$. Then the real measure of real inventory change Q_t^*, defined as,

$$Q_t^* = Q_{t-1}^* + \mu_t \mathbf{PA}_{t-1} \cdot (\mathbf{q}_t - \mathbf{q}_{t-1}) + (1-\mu_t) \frac{\mathbf{PA}_t \cdot (\mathbf{q}_t - \mathbf{q}_{t-1})}{L_t^{ds}}, \tag{22}$$

satisfies axioms I and IV, and approximately satisfies axioms II and III.

PROOF: See appendix.

To generalize equation (22) to any time period, let CF_s^{ds} be the chained Fisher index for the arbitrary period *s* constructed from the dominant-sign Laspeyres and Paasche indexes. Then:

$$Q_s^* = Q_{s-1}^* + \mu_s \left[\frac{\mathbf{PA}_{t-1}}{CF_{s-1}^{ds}} + \frac{\mathbf{PA}_s}{CF_s^{ds}}\right] \cdot (\mathbf{q}_s - \mathbf{q}_{s-1}) \tag{23}$$

Assuming that period *t*–1 is the base period, when prices are all normalized to 1, Q_{t-1}^* in equation (22) is a simple sum of fixed-price CIIs. Equation (21) measures real CII in period *t* in terms of prices of period t–1. We can also use the difference equation (23) to estimate real CII in any time period by cumulating changes from the base year.

As an alternative to calculating weights for the Laspeyres and Paasche price indexes by zeroing out each q_{it} that disagrees in sign with $\mathbf{PA}_t \cdot \mathbf{q}_t$, and similarly for each $q_{i,t\text{-}1}$ that disagrees in sign with $\mathbf{PA}_{t\text{-}1} \cdot \mathbf{q}_{t\text{-}1}$, we could use absolute values of the q_{it} to weight the price index.

Figure 5 shows how the proposed measure of real CII with price index weights based on absolute values of $\mathbf{q}_t$ performs when applied to annual data of the sort used in the NIPAs. Figure 6 repeats the test of the proposed method using quarterly data. Despite the existence of hypothetical examples of illogical behavior of the current measure of real CII based on the change in Fisher stocks, the performance of this measure with actual data is similar to the performance of the theoretically superior method that uses dominant sign flows as weights in the deflator. Nevertheless, some downward bias in the slope of the long run trend seems to be present in the difference-of-stocks method. Theory suggests that such a bias is to be expected: the user cost formula for the inventory stocks implies that this cost is high when the stockpiled item has a declining price and low when the stockpiled item has a rising price. Businesses' desire to avoid holding losses on inventories and to benefit from holding gains may therefore make the stock weights high for the items with the largest price increases, imparting an upward bias to the deflator and a downward bias to the measure of real CIPI based on first differences of real stocks.

Figure 5: Real Nonfarm Change in Private Inventories, Annual Data (in millions of chained 2000 dollars)

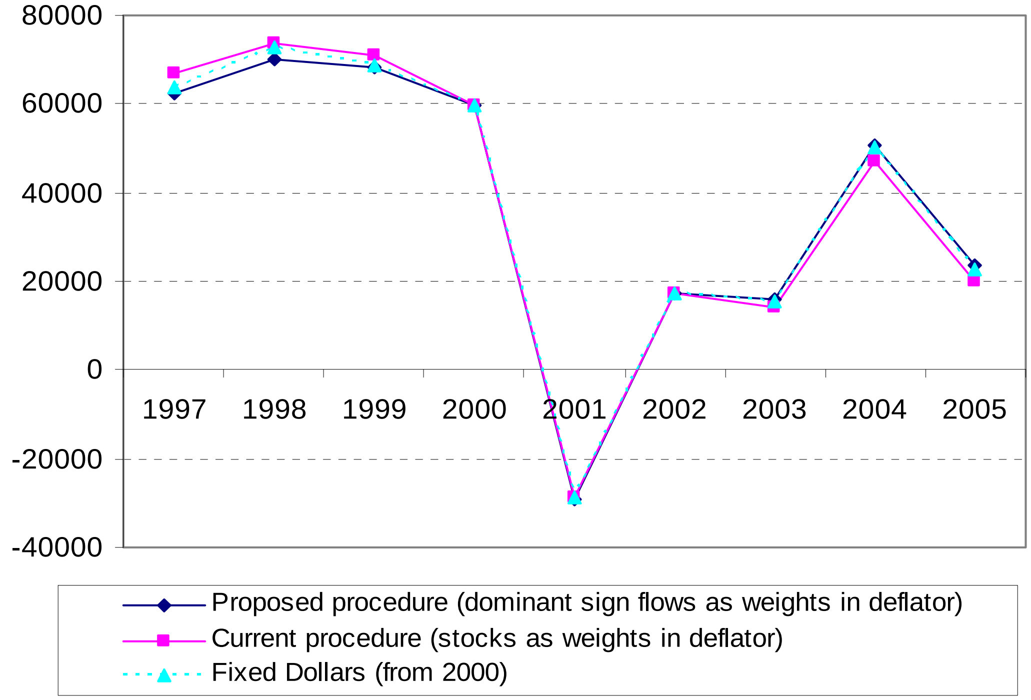
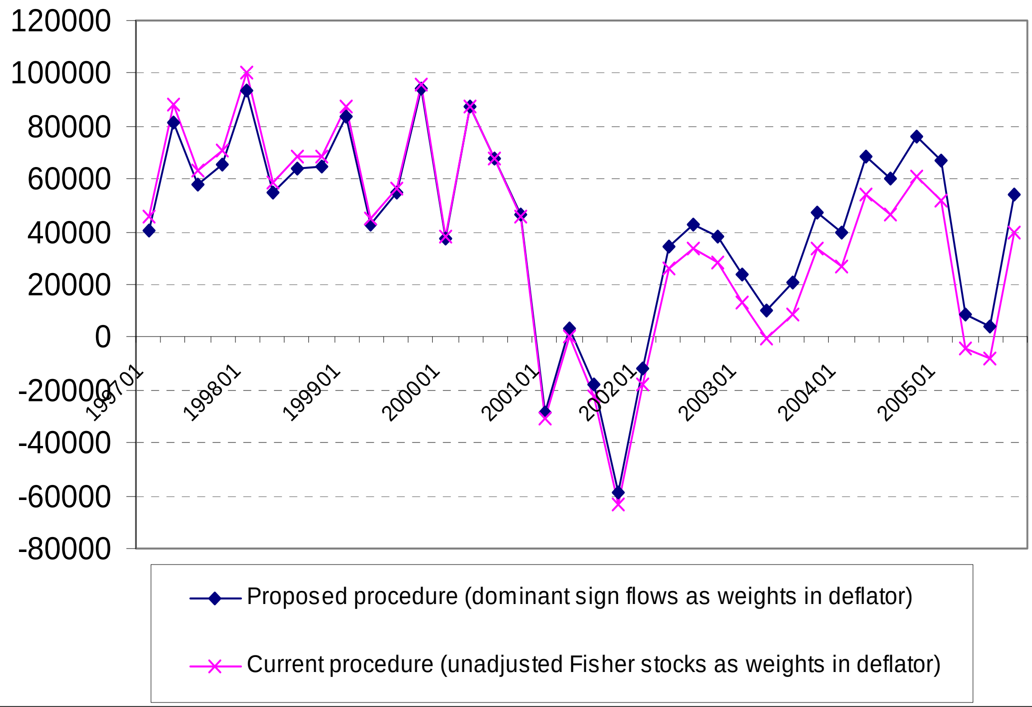


Figure 6: Real Nonfarm Change in Private Inventories, Quarterly Data (in Millions of Chained 2000 Dollars)



IV. Conclusion

Procedures that are theoretically correct can work poorly in actual practice when measurement error is present in the data. (The problem of formula bias in the US CPI, which was caused by an attempt to implement a theoretically elegant sample of estimator of a Laspeyres price index without taking full account of the practical difficulties in measuring weights, is an example of this.) In the case of nominal inventory investment for a year, the theoretically correct procedure sums the values for the months within the year even though the inventory additions and withdrawals in those months may take place at different prices. This has the effect of incorporating a potentially large adjustment for holding gains and losses on inventories in the calculation of annual production as annual final purchases plus annual net inventory investment. However, a large adjustment of this type should be avoided unless we are certain that the data require it, and a zero adjustment is most advantageous. Moreover, given the likelihood of at least some inaccuracy in the monthly data, we cannot hope to measure holding gains and losses on inventories with much precision. The procedure that we develop for measuring nominal inventory investment incorporates an adjustment of zero for these holding gains and losses if that is not too far from the one implied by the pattern of the monthly data. Otherwise, the adjustment is damped towards zero.

For the problem of measuring real inventory investment, the solution that is most satisfactory in theory is to say that the presence of sign changes prevents the construction of the index numbers necessary to define the concept. In practice, however, real inventory investment is a valuable statistic to users of national accounts, and the data are often well-behaved enough to make even a conventional measure of this concept informative. Suppressing this statistic is therefore not a practical alternative. The commonly used procedure of first differencing the

Fisher inventory stocks is one possible solution to real inventory investment problem. Yet this procedure can yield distorted results because it implicitly uses inventory *stocks* as weights in a price index that deflates the *flows*. In this paper, we develop a measure of real inventory investment based on cumulative deflated differences. Price indexes weighted by quantity flows of the dominant sign are used to deflate the differences. A test of the method of the method implies a similar pattern to the first difference of stocks, except for a flattening out of the downward tilt of the long run profile. Theory suggests that this downward tilt reflects a downward bias that is avoided by the use of flow-based weights to calculate a deflator for the flows.

References

Commission of the European Communities, International Monetary Fund, Organisation for Economic Co-operation and Development, United Nations, and World Bank, 1993. *System of National Accounts 1993*, Brussels/Luxembourg, New York, Paris, Washington.

Diewert, W. Erwin. 2004. “Index Number Problems in the Measurement of Real net Exports and Real net Changes in Inventories.” Manuscript, Univ. of British Columbia.

Diewert, W. Erwin. 2005. “On Measuring Inventory Change in Current and Constant Dollars.” Discussion Paper No. 05-12, Department of Economics, University of British Columbia.

Ehemann, Christopher. 2005. An Alternative Estimate of Real Inventory Change for National Economic Accounts. *International Journal of Production Economics* 93-94, pp.101-110.

Foss, Murray F., Gary Fromm, and Irving Rottenberg. (no date.) *Measurement of Business Inventories*, US Dept. of Commerce Economic Research Report 3.

Hicks, John R. 1940. *The Valuation of Social Income*, Economica 7 (May): 105-124

Lal, Kishori. 2001. “Aggregating Sub-Annual Current Price Value of Changes inInventories to Annual Totals under Condition of Inflation—An Inherent Dilemma.” Manuscript, Statistics Canada.

United Nations, 1993. See Commission of the European Communities, *et al.*

Table 1: Analysis of Actual Data on Inventories for Barley in Canada, 1988*

	Q1	Q2	Q3	Q4	Year (total of quarters)
Physical Change (tons)	-3,408,979	-3,268,099	5,144,839	-369,812	**-1,902,051**
Value of Physical Change ($ thousands)	-192,766	-246,795	564,428	-37,326	**87,541******
Implicit Average Price	$56.55	$75.52	$109.71	$100.93	**–$46.02**
Annual Average Price					**$85.68**
Quarterly change at average annual price ($ thousands)	-292,067	-279,997	440,788	-31,684	**–162,960**

* Based on Baldwin (2006), table 2, which came from a memo by Richard T. Richards and Nugent Miller of Statistics Canada.

** This is the SNA concept for nominal annual change in inventories.

Table 2: Possible Ways to Estimate Nominal Production of Barley in 1988

(thousands of dollars)

1. Value CII at average consumption price and include a revaluation term	
Consumption (value of gross inventory withdrawals in Q1, Q2 and Q4)	**476,887**
LESS: Consumption supplied by past production (net CII at average consumption price)	128,718
EQUALS: Consumption of current production valued at contemporaneous prices	348,169
PLUS: Revaluation from average consumption price ($67.67) to production price ($100.93)	216,259
EQUALS: **Current production at contemporaneous price**	**564,428**
MEMO: Holding gain on inventories	*–216,259*
2. SNA Method	
Consumption	**476,887**
PLUS: Net CII at contemporaneous prices	87,541
EQUALS: Production at contemporaneous prices	564,428
3. Value CII at the average annual price	
Net CII at average annual price ($85.68)	–162,960
Estimate of production as consumption + CII at average annual price	313,927
Price implicitly used to value current production by the above	*$61.02*

Table 3: Benchmarking Procedure Illustrated with data on Barley Inventories

	Q1	Q2	Q3	Q4	Year
Revised physical Change (tons)	-2,043,637	-2,791,910	4,018,417	-1,084,921	**-1,902,051**
Value of revised physical change ($1000s)	-$115,561	-$210,835	$440,851	-$109,504	**$4,952**
Implicit Average Price	$56.55	$75.52	$109.71	$100.93	
Annual physical consumption (tons)					**7,046,890**
Annual physical consumption + physical change in inventories (tons)					**5,144,839**
Annual nominal consumption					**$476,887**
Annual nominal consumption + revised nominal CII					**$481,839**
Implicit average price of production					**$93.65**

Appendix

1. Derivation of Equation (6)

Let $\hat{\beta}_i$ denote $\frac{\mathrm{Cov}(p_{it},q_{it})}{\mathrm{Var}(p_{it})}$, the regression slope coefficient. Then

$$\tilde{q}_{it} = q_i^A/12 + q_{it}^M - [\,\bar{q}_i^M + k\hat{\beta}_i(p_{it} - \bar{p}_i)\,].$$

The variance of $\tilde{q}_{it}$ relative to the variance of q_{it}^M is:

$$\frac{\mathrm{Var}(\tilde{q}_{it})}{\mathrm{Var}(q_{it}^M)} = 1 - 2k\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)} + k^2\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)}$$

Let $1 - \mathrm{R}^{2*}$ represent the proportion of the variance that remains after benchmarking, or the ratio of $\mathrm{Var}(\tilde{q}_{it})$ to $\mathrm{Var}(q_{it}^M)$. Then:

$$1 - \mathrm{R}^{2*} = 1 - 2k\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)} + k^2\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)}$$

$$0 = \mathrm{R}^{2*} - 2k\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)} + k^2\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)}$$

From the quadratic formula we have:

$$k = \left[2\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)} - \left\{\left[2\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)}\right]^2 - 4\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)}\mathrm{R}^{2*}\right\}^{1/2}\right] \Big/ \left[2\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)}\right]$$

$$k = 1 - \left\{1 - \mathrm{R}^{2*}/\left(\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)}\right)\right\}^{1/2}$$

But

$$\hat{\beta}_i^2\frac{\mathrm{Var}(p_{it})}{\mathrm{Var}(q_{it}^M)} = \frac{\mathrm{Cov}(p_{it},q_{it})^2}{\mathrm{Var}(p_{it})\mathrm{Var}(q_{it}^M)} = \mathrm{R}^2$$

so,

$$\mathrm{k} = 1 - \{1 - \mathrm{R}^{2*}/\mathrm{R}^2\}^{1/2}.$$

2. Proof of Proposition 2

Axiom I makes the assumption that $\mathbf{q}_t \geq 0$, which implies that $\mathbf{a}_t = \mathbf{q}_t$. The right side of equation (21) can then be simplified to $Q^*_{t-1}F^q_t$ as follows:

$$Q^*_{t-1} + \mu_t \mathbf{PA}_{t-1}\cdot(\mathbf{q}_t - \mathbf{q}_{t-1}) + (1-\mu_t)\frac{\mathbf{PA}_t\cdot(\mathbf{q}_t - \mathbf{q}_{t-1})}{L^a_t}$$

$$= \mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1} + \mu_t \mathbf{PA}_{t-1}\cdot(\mathbf{q}_t - \mathbf{q}_{t-1}) + (1-\mu_t)\frac{\mathbf{PA}_t\cdot(\mathbf{q}_t - \mathbf{q}_{t-1})}{L^a_t}$$

$$= \mu_t \mathbf{PA}_{t-1}\cdot\mathbf{q}_t + (1-\mu_t)\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{L^a_t}$$

$$= \frac{1}{1+[L^a_t/P^a_t]^{0.5}}\mathbf{PA}_{t-1}\cdot\mathbf{q}_t + \frac{1}{1+[P^a_t/L^a_t]^{0.5}}\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{L^a_t}$$

$$= \mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}\left\{\frac{[P^a_t]^{0.5}}{[P^a_t]^{0.5}+[L^a_t]^{0.5}}\frac{\mathbf{PA}_{t-1}\cdot\mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}} + \frac{[L^a_t]^{0.5}}{[P^a_t]^{0.5}+[L^a_t]^{0.5}}\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{\mathbf{PA}_t\cdot\mathbf{q}_{t-1}}\right\}$$

$$= \frac{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}{[P^a_t]^{0.5}+[L^a_t]^{0.5}}\left\{\left[\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_t}\right]^{0.5}\frac{\mathbf{PA}_{t-1}\cdot\mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}} + \left[\frac{\mathbf{PA}_t\cdot\mathbf{q}_{t-1}}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}\right]^{0.5}\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{\mathbf{PA}_t\cdot\mathbf{q}_{t-1}}\right\}$$

$$= \frac{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}{[P^a_t]^{0.5}+[L^a_t]^{0.5}}\left\{\left[\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}\right]^{0.5}\left[\frac{\mathbf{PA}_{t-1}\cdot\mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}\right]^{0.5} + \left[\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}\right]^{0.5}\left[\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{\mathbf{PA}_t\cdot\mathbf{q}_{t-1}}\right]^{0.5}\right\}$$

$$= \frac{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}{[P^a_t]^{0.5}+[L^a_t]^{0.5}}\left\{\left[\frac{\mathbf{PA}_t\cdot\mathbf{q}_{t-1}}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}\right]^{0.5}\left[\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{\mathbf{PA}_t\cdot\mathbf{q}_{t-1}}\right]^{0.5}\left[\frac{\mathbf{PA}_{t-1}\cdot\mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}\right]^{0.5}\right.$$

$$\left. + \left[\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_t}\right]^{0.5}\left[\frac{\mathbf{PA}_{t-1}\cdot\mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot\mathbf{q}_{t-1}}\right]^{0.5}\left[\frac{\mathbf{PA}_t\cdot\mathbf{q}_t}{\mathbf{PA}_t\cdot\mathbf{q}_{t-1}}\right]^{0.5}\right\}$$

$$= \frac{\mathbf{PA}_{t-1}\cdot \mathbf{q}_{t-1}}{[P^a_t]^{0.5} + [L^a_t]^{0.5}} \Big\{ [L^a_t]^{0.5}\left[\frac{\mathbf{PA}_t\cdot \mathbf{q}_t}{\mathbf{PA}_t\cdot \mathbf{q}_{t-1}}\right]^{0.5}\left[\frac{\mathbf{PA}_{t-1}\cdot \mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot \mathbf{q}_{t-1}}\right]^{0.5}$$

$$+ \; [P^a_t]^{0.5}\left[\frac{\mathbf{PA}_{t-1}\cdot \mathbf{q}_t}{\mathbf{PA}_{t-1}\cdot \mathbf{q}_{t-1}}\right]^{0.5}\left[\frac{\mathbf{PA}_t\cdot \mathbf{q}_t}{\mathbf{PA}_t\cdot \mathbf{q}_{t-1}}\right]^{0.5}\Big\}$$

$$= \mathbf{PA}_{t-1}\cdot \mathbf{q}_{t-1}\, F^q_t .$$

The assumption in the approximate version of axiom II is that $\mathbf{q}_t'$ implies a value for real GDP that exceeds the value for real GDP implied by $\mathbf{q}_t'$ by at least some $\Delta > 0$. This assumption may be written as $Q^{GDP'}_t - Q^{GDP}_t \geq \Delta$, where $Q^{GDP'}_t$ denotes the value of equation (1) with $\mathbf{q}_t'$ substituted for $\mathbf{q}_t$. This implies that,

$$\left[\frac{\mathbf{p}^f_{t-1}\cdot \mathbf{q}^f_t + \mathbf{PA}_{t-1}\cdot \mathbf{q}_t'}{\mathbf{p}^f_{t-1}\cdot \mathbf{q}^f_{t-1} + \mathbf{PA}_{t-1}\cdot \mathbf{q}_{t-1}}\,\frac{\mathbf{p}^f_t\cdot \mathbf{q}^f_t + \mathbf{PA}_t\cdot \mathbf{q}_t'}{\mathbf{p}^f_t\cdot \mathbf{q}^f_{t-1} + \mathbf{PA}_t\cdot \mathbf{q}_{t-1}}\right]^{0.5} - \left[\frac{\mathbf{p}^f_{t-1}\cdot \mathbf{q}^f_t + \mathbf{PA}_{t-1}\cdot \mathbf{q}_t}{\mathbf{p}^f_{t-1}\cdot \mathbf{q}^f_{t-1} + \mathbf{PA}_{t-1}\cdot \mathbf{q}_{t-1}}\,\frac{\mathbf{p}^f_t\cdot \mathbf{q}^f_t + \mathbf{PA}_t\cdot \mathbf{q}_t}{\mathbf{p}^f_t\cdot \mathbf{q}^f_{t-1} + \mathbf{PA}_t\cdot \mathbf{q}_{t-1}}\right]^{0.5}$$

$$\geq \frac{\Delta}{\mathbf{p}^f_{t-1}\cdot \mathbf{q}^f_{t-1} + \mathbf{PA}_{t-1}\cdot \mathbf{q}_{t-1}} . \qquad \text{(A-1)}$$

Let $\theta_1 = \mathbf{PA}_{t-1}\cdot \mathbf{q}_t' - \mathbf{PA}_{t-1}\cdot \mathbf{q}_t$ and $\theta_2 = \mathbf{PA}_t\cdot \mathbf{q}_t' - \mathbf{PA}_t\cdot \mathbf{q}_t$. Let L^P denote the Laspeyres price for GDP, let $L^Q(\mathbf{q}_t)$ denote the Laspeyres quantity index for GDP evaluated at $\mathbf{q}_t$, and let $P^Q(\mathbf{q}_t)$ and $F^Q(\mathbf{q}_t)$ be the analogous Paasche and Fisher indexes. Inequality (A-1) implies that θ_1 and θ_2 satisfy the condition that:

$$\Big[[L^Q(\mathbf{q}_t) + \theta_1/GDP_{t-1}][P^Q(\mathbf{q}_t) + \theta_2/(L^P GDP_{t-1})]\Big]^{0.5} \geq \Delta/GDP_{t-1} + F^Q(\mathbf{q}_t) \qquad \text{(A-2)}$$

Squaring, and then canceling out $L^Q(\mathbf{q}_t)P^Q(\mathbf{q}_t)$ on the left with $[F^Q(\mathbf{q}_t)]^2$ on the right gives:

$$\theta_1 P^Q(\mathbf{q}_t)/\text{GDP}_{t-1} + \theta_2 L^Q(\mathbf{q}_t)/(\text{GDP}_{t-1})L^P + \theta_1\theta_2/(\text{GDP}_{t-1})^2 L^P \geq (\Delta/\text{GDP}_{t-1})^2 + 2F^Q(\mathbf{q}_t)(\Delta/\text{GDP}_{t-1}) \quad \text{(A-3)}$$

The terms that are divided by GDP squared are small enough to be ignored, so the requirement that the change in CIPI raise real GDP by at least Δ is effectively a requirement that a weighted average of θ_1 and a deflated value of θ_2 exceed Δ times a constant of proportionality approximately equal to 1:

$$\frac{P^Q(\mathbf{q}_t)}{P^Q(\mathbf{q}_t) + L^Q(\mathbf{q}_t)}\theta_1 + \frac{L^Q(\mathbf{q}_t)}{P^Q(\mathbf{q}_t) + L^Q(\mathbf{q}_t)}\theta_2/L^P \geq \frac{2F^Q(\mathbf{q}_t)}{P^Q(\mathbf{q}_t) + L^Q(\mathbf{q}_t)}\Delta. \quad \text{(A-4)}$$

The expression for real CIPI in equation (9) implies that its change is also proportional to a weighted average of θ_1 and a deflated value of θ_2:

$$Q^*_t - Q^*_{t-1} = \frac{[P^a_t]^{0.5}}{[P^a_t]^{0.5} + [L^a_t]^{0.5}}\theta_1 + \frac{[L^a_t]^{0.5}}{[P^a_t]^{0.5} + [L^a_t]^{0.5}}\theta_2/L^a_t. \quad \text{(A-5)}$$

Solving (A-4) for θ_2 as a function of θ_1 and Δ, then substituting the resulting expression for θ_2 in equation (A-5) and substituting F^a_t for $[L^a_t P^a_t]^{0.5}$ implies:

$$[Q^*_t - Q^*_{t-1}]\{F^a_t + L^a_t\} \geq \frac{2F^Q(\mathbf{q}_t)L^P}{L^Q(\mathbf{q}_t)}\Delta - \frac{P^Q(\mathbf{q}_t)L^P}{L^Q(\mathbf{q}_t)}\theta_1 + [F^a_t]^{0.5}\theta_1 \quad \text{(A-6)}$$

Substituting P^P, the Paasche price index for GDP, for $\frac{P^Q(\mathbf{q}_t)L^P}{L^Q(\mathbf{q}_t)}$, the right side of inequality (A-6) will be positive, implying that $Q^*_t - Q^*_{t-1}$ is positive, if Δ exceeds a value that is proportional the difference between the Paasche price index for GDP and the absolute-value weighted Fisher price index for change in inventories:

$$\Delta \geq \left[P^P - F_t^a\right]\frac{L^Q(\mathbf{q}_t)}{2F^Q(\mathbf{q}_t)L^P}\theta_1 \tag{A-7}$$

In general, the difference between the price indexes in (A-7) is likely to be small, implying that a small positive effect of inventory change on real GDP is a sufficient condition for an increase in real CIPI.

An approximate version of axiom III, sign agreement, follows from the approximate satisfaction of axiom II by letting $\mathbf{q}_t = \mathbf{q}_{t-1}$.

Axiom IV is satisfied because multiplying every element of $\mathbf{PA}_t$ by the same positive scalar will not affect the ratio of the Laspeyres and Paasche price indexes or the deflated value of $\mathbf{PA}_t$, given by $\frac{\mathbf{PA}_t}{L_t^a}$. □